ON LINEAR EXTENSION FOR INTERPOLATING SEQUENCES.

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Abstract. Let A be a uniform algebra on the compact space X and σ a probability measure on X. We define the Hardy spaces $H^p(\sigma)$ and the $H^p(\sigma)$ interpolating sequences S in the p-spectrum $\mathcal{M}_p$ of σ. We prove, under some structural hypotheses on σ that "Carleson type" conditions on S imply that S is interpolating with a linear extension operator in $H^s(\sigma),\ s<p$ provided that either $p=\infty$ or $p\leq 2$.

This gives new results on interpolating sequences for Hardy spaces of the ball and the polydisc. In particular in the case of the unit ball of $\mathbb{C}^n$ we get that if there is a sequence $\{\rho_a\}_{a\in S}$ bounded in $H^\infty(\mathbb{B})$ such that $\forall a,b\in S,\ \rho_a(b)=\delta_{ab}$, then S is $H^p(\mathbb{B})$-interpolating with a linear extension operator for any $1\leq p<\infty$.

1. Introduction

Let $\mathbb{B}$ be the unit ball of $\mathbb{C}^n$; we denote as usual by $H^p(\mathbb{B})$ the Hardy spaces of holomorphic functions in $\mathbb{B}$. Let S a sequence of points in $\mathbb{B}$ and $1\leq p\leq\infty$; we say that S is H^p-interpolating if

$$\forall\lambda\in\ell^p(S),\ \exists f\in H^p(\mathbb{B})\ s.t.\ \forall a\in S,\ f(a)=\lambda_a(1-|a|^2)^{n/p}.$$

Let $a\in\mathbb{B}$ we set $k_a(z):=\dfrac{1}{(1-\overline{a}\cdot z)^n}$ its reproducing kernel and $k_{p,a}:=\dfrac{k_a}{\|k_a\|_p}$ the normalized reproducing kernel for a in $H^p(\mathbb{B})$. Now if S is H^p-interpolating, then we have, with p' the conjugate exponent for p:

$$\exists C>0,\ \forall a\in S,\ \exists\rho_a\in H^p(\mathbb{B})\ s.t.\ \langle\rho_a,\ k_{p',b}\rangle=\delta_{ab}.$$

We shall say that S is dual bounded in $H^p(\mathbb{B})$ if the dual system $\{\rho_a\}_{a\in S}$ to $\{k_{p',a}\}_{a\in S}$ exits and is bounded in $H^p(\mathbb{B})$.

Hence if S is H^p-interpolating then S is dual bounded in $H^p(\mathbb{B})$.

Definition 1.1. *We say that the $H^p(\mathbb{B})$ interpolating sequence S has the linear extension property (LEP) if there is a bounded linear operator $E\ :\ \ell^p\longrightarrow H^p(\mathbb{B})$ such that $\forall\lambda\in\ell^p,\ E\lambda$ interpolates the sequence λ in $H^p(\mathbb{B})$ on S, i.e.*

$$\exists C>0,\ \forall\lambda\in\ell^p,\ E\lambda\in H^p(\mathbb{B}),\ \|E\lambda\|_p\leq C\ s.t.\ \forall a\in S,\ E\lambda(a)=\lambda_a\|k_a\|_{p'}$$

Natural questions are the following:

If S is dual bounded in $H^p(\mathbb{B})$, is $S\in IH^p(\mathbb{B})$?

If $S\in IH^p(\mathbb{B})$ has S automatically the LEP ?

This is true in the classical case of the Hardy spaces of the unit disc $\mathbb{D}$:

for $p=\infty$ this is the famous characterization of H^∞ interpolating sequences by L. Carleson [7] and the LEP was given by P. Beurling [6].

for $p \in [1,\ \infty[$ this was done by H. Shapiro and A. Shields [16] and because the characterization is the same for all $p \in [1, \infty]$, the LEP is deduced easily from the H^∞ case and was done explicitly with $\overline{\partial}$ methods in [2].

For the Bergman classes $A^p(\mathbb{D})$, it is no longer true that the interpolating sequences are the same for $A^p(\mathbb{D})$ and $A^q(\mathbb{D})$, $q \neq p$. But A.P. Schuster and K. Seip [15], [14] proved that S dual bounded in $A^p(\mathbb{D})$ implies S $A^p(\mathbb{D})$-interpolating still with the LEP.

The first question is open, even in the ball $\mathbb{B}$ of $\mathbb{C}^n$, $n \geq 2$, with $H^p(\mathbb{B})$, the usual Hardy spaces of the ball or in the polydisc $\mathbb{D}^n$ of $\mathbb{C}^n$, $n \geq 2$ still with the usual Hardy spaces.

The second one is known only in the case $p = \infty$ as we shall see later.

Nevertheless in the case of the unit ball of $\mathbb{C}^n$, B. Berndtsson [4] proved that if the product of the Gleason distances of the points of S is bounded below away of 0 then S is $H^\infty(\mathbb{B})$. He also proved that this condition is not necessary for $n > 1$.

B. Berndtsson, A. S-Y. Chang and K-C. Lin [5] proved the same theorem in the polydisc of $\mathbb{C}^n$.

In this paper we shall prove that loosing a little bit on the value of p, S dual bounded in $H^p(\mathbb{B})$ implies $\forall s < p,\ S \in IH^s(\mathbb{B})$ with the LEP, provided that $1 < p \leq 2$ or $p = \infty$. In particular:

Theorem 1.2. *If $S \subset \mathbb{B}$ is dual bounded in $H^p(\mathbb{B})$, then it is H^s-interpolating for any $1 \leq s < p$, provided that $p \in]1, 2]$ or $p = \infty$. Moreover S has the property that there is a bounded linear operator from $\ell^s(S) \longrightarrow H^s(\mathbb{B})$ doing the interpolation.*

The methods we use being purely functional analytic, these results extend to the setting of uniform algebras.

2. Uniform algebras.

Let A be a uniform algebra on the compact space X, i.e. A is a sub-algebra of $\mathcal{C}(X)$, the continuous functions on X, which separates the points of X and contains 1.

Let σ be a probability measure on X.

For $1 \leq p < \infty$ we define as usual the Hardy space $H^p(\sigma)$ as the closure of A in $L^p(\sigma)$.
$H^\infty(\sigma)$ will be the weak-* closure of A in $L^\infty(\sigma)$.

Let $\mathcal{M}$ be the Guelfand spectrum of A, i.e. the multiplicative elements of A'. We note the same way an element of A and its Guelfand transform:

$$\forall a \in \mathcal{M} \subset A',\ \forall f \in A,\ f(a) := \hat{f}(a) = a(f).$$

We shall use the following notions, already introduced in [3].

Definition 2.1. *Let $\mathcal{M}$ be the spectrum of A and $a \in \mathcal{M}$; we call $k_a \in H^p(\sigma)$ a p-reproducing kernel for the point a if $\forall f \in A,\ f(a) = \int_X f(\zeta)\overline{k}_a(\zeta)\, d\sigma(\zeta)$.*

We define the p-spectrum of σ as the subset $\mathcal{M}_p$ of $\mathcal{M}$ such that every element has a p'-reproducing kernel with p' the conjuguate exponent for p, $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 2.2. *We say that $S \subset \mathcal{M}_p$ is $H^p(\sigma)$ interpolating for $1 \leq p < \infty$, $S \in IH^p(\sigma)$ if*

$$\forall \lambda \in \ell^p,\ \exists f \in H^p(\sigma)\ s.t.\ \forall a \in S,\ f(a) = \lambda_a \left\| k_a \right\|_{p'}.$$

We say that $S \subset \mathcal{M}_\infty$ is $H^\infty(\sigma)$ interpolating, $S \in IH^\infty(\sigma)$ if

$$\forall \lambda \in \ell^\infty,\ \exists f \in H^\infty(\sigma)\ s.t.\ \forall a \in S,\ f(a) = \lambda_a.$$

Remark 2.3. *If S is $H^p(\sigma)$-interpolating then there is a constant C_I, the interpolating constant, such that* [3]*:*

$$\forall \lambda \in \ell^p,\ \exists f \in H^p(\sigma),\ \left\| f \right\|_p \leq C_I \left\| \lambda \right\|_p,\ s.t.\ \forall a \in S,\ f(a) = \lambda_a \left\| k_a \right\|_{p'}.$$

Definition 2.4. *We say that the $H^p(\sigma)$ interpolating sequence S has the linear extension property (LEP) if there is a bounded linear operator $E\ :\ \ell^p \longrightarrow H^p(\sigma)$ such that $\forall \lambda \in \ell^p,\ E\lambda$ interpolates the sequence λ in $H^p(\sigma)$ on S, i.e.*

$$\exists C > 0,\ \forall \lambda \in \ell^p,\ E\lambda \in H^p(\sigma),\ \|E\lambda\|_p \leq C \ s.t.\ \forall a \in S,\ E\lambda(a) = \lambda_a \|k_a\|_{p'}$$

Let $S \subset \mathcal{M}_p$, so $k_{p',a} := \dfrac{k_a}{\|k_a\|_{p'}}$, the normalized reproducing kernel, exits for any $a \in S$; let us consider a dual system $\{\rho_a\}_{a\in S} \subset H^p(\sigma)$, i.e. $\forall a, b \in S,\ \langle \rho_a,\ k_{p',b}\rangle = \delta_{a,b}$ when it exists.

Definition 2.5. *We say that $S \subset \mathcal{M}_p$ is dual bounded in $H^p(\sigma)$ if a dual system $\{\rho_a\}_{a\in S} \subset H^p(\sigma)$ exists and if this sequence is bounded in $H^p(\sigma)$, i.e. $\exists C > 0$ s.t. $\forall a \in S,\ \|\rho_a\|_p \leq C$.*

We shall show that, under some structural hypotheses on σ and the fact that S is Carleson (the definition of Carleson sequences will be given later):

Theorem 2.6. *If $1 \leq s < p$ and either $p \leq 2$ or $p = \infty$, $S \subset \mathcal{M}_p \cap \mathcal{M}_s$ is dual bounded in $H^p(\sigma)$ and S is a Carleson sequence, then $S \in IH^s(\sigma)$ with the linear extension property.*

The passage from $p = 2$ to $p \leq 2$ in the case of the ball is due to F. Bayart: he uses Khintchine's inequalities which reveal to be very well fitted to this problem. In fact F. Lust-Piquart showed me a way not to use Khintchine's inequalities: one can use the fact that L^p spaces are of type p in the part $p \leq 2$ in the proof of theorem 2.6.

I shall add this proof.

The case $p = \infty$ of this theorem is the best possible in this generality. There is no hope to have that dual boundedness in H^∞ implies H^∞-interpolation as L. Carleson proved for the unit disc.

In [10] and in [12] the authors proved that in the spectrum of the uniform algebra $H^\infty(\mathbb{D})$ there are sequences S of points such that the product of the Gleason distances is bounded below away from 0, which implies that S is dual bounded in $H^\infty(\mathbb{D})$, but S is *not* H^∞-interpolating.

The general theorem 2.6 implies a polydisc and a ball version.
In the polydisc $\mathbb{D}^n \subset \mathbb{C}^n$ the structural hypotheses are true [3], hence

Theorem 2.7. *Let $S \subset \mathbb{D}^n$ be a Carleson sequence and dual bounded in $H^p(\mathbb{D}^n)$ with either $p = \infty$ or $1 < p \leq 2$, then S is $H^s(\mathbb{D}^n)$ interpolating for any $1 \leq s < p$ with the LEP.*

In the ball, the structural hypothesesare true [3] and moreover we know, by an easy corollary of a theorem of P. Thomas [18], that S dual bounded in $H^p(\mathbb{B})$ implies S Carleson, hence

Theorem 2.8. *Let $S \subset \mathbb{B}$ be dual bounded in $H^p(\mathbb{B})$ with either $p = \infty$ or $1 < p \leq 2$, then S is $H^s(\mathbb{B})$ interpolating for any $1 \leq s < p$ with the LEP.*

As usual by use of the "subordination lemma" [1] we have the same result for the Bergman classes of the ball. Denote by $A^p(\mathbb{B})$ the holomorphic functions in $L^p(\mathbb{B})$ for the area measure of the ball then

Corollary 2.9. *Let $S \subset \mathbb{B}$ be dual bounded in $A^p(\mathbb{B})$ with either $p = \infty$ or $1 < p \leq 2$, then S is $A^s(\mathbb{B})$ interpolating for any $1 \leq s < p$ with the LEP.*

In [3] it was proved:

Theorem 2.10. *Let $p \geq 1$, $1 \leq s < p$ and q be such that $\dfrac{1}{s} = \dfrac{1}{p} + \dfrac{1}{q}$. Suppose that $S \subset \mathcal{M}_s \cap \mathcal{M}_{q'}$ is $H^p(\sigma)$ interpolating, q-Carleson and σ verifies the structural hypotheses , then S is $H^s(\sigma)$ interpolating.*

The theorem 2.10 is better for $p \in [1,\ 2]$ or $p = \infty$: we have the LEP under the weaker assumption that S is dual bounded in $H^p(\sigma)$.

But we have not the full range of p as in theorem 2.10.

3. Reproducing kernels.

Let us recall some facts about reproducing kernels and p-spectrum.

First the reproducing kernel for $a \in \mathcal{M}$ if it exists is unique. Suppose there are 2 of them, $k_a \in H^p(\sigma)$ and $k'_a \in H^q(\sigma)$:

$$\forall f \in A,\ 0 = f(a) - f(a) = \int_X f(\overline{k}_a - \overline{k}'_a)\, ds \Longrightarrow k_a = k'_a\ \sigma - a.e.$$

because, by definition, A is dense in $H^r(\sigma)$ with $r := \min(p, q)$. Hence it is correct to denote it by k_a without reference to the $H^p(\sigma)$ where it belongs.

Let $a \in \mathcal{M}_p$ then $k_a \in H^{p'}(\sigma)$; if $p < q \Longrightarrow q' < p'$ hence $k_a \in H^{q'}(\sigma)$ because σ is a probability measure so $a \in \mathcal{M}_q$ and we have $p < q \Longrightarrow \mathcal{M}_p \subset \mathcal{M}_q$.

To simplify the notation we shall use:

$$\langle f, g\rangle := \int_X f\overline{g}\, d\sigma,$$

whenever this is meaningfull.

If $a \in \mathcal{M}_2$ we always have a "Poisson kernel" associated to a, $P_a := \dfrac{|k_a|^2}{\|k_a\|_2^2}$ and the well known

Lemma 3.1. *$P_a \in L^1(\sigma),\ \|P_a\|_1 = 1$ and*

$$\forall f \in A,\ f(a) = \langle f,\ P_a\rangle = \int_X f P_a\, d\sigma.$$

Proof

$$\int_X f P_a\, d\sigma = \int_X f \frac{k_a \overline{k}_a}{\|k_a\|_2^2}\, d\sigma = \frac{1}{\|k_a\|_2^2} f(a) k_a(a) = f(a),$$

because $f k_a \in H^2(\sigma)$ and $k_a(a) = \int_X k_a \overline{k}_a\, d\sigma = \|k_a\|_2^2$. □

This allows us to define the Poisson integral of a bounded function on X:

Definition 3.2. *Let $f \in L^\infty(\sigma)$ we set $\forall a \in \mathcal{M}_2,\ \tilde{f}(a) := \langle f,\ P_a\rangle$ its Poisson integral. If $f \in L^2(\sigma)$ we set $f^* := P_2 f$ its orthogonal projection on $H^2(\sigma)$; we extend f^* on $\mathcal{M}_2$:*

$$\forall f \in L^2(\sigma),\ \forall a \in \mathcal{M}_2,\ f^*(a) := \langle f^*, k_a\rangle = \langle f,\ k_a\rangle .$$

Of course if $f \in A$ we have $f^* = \tilde{f} = f$ and for any $f \in L^\infty(\sigma)$, $\widetilde{(f^*)} = f^*$.

3.1. Structural hypotheses.

We shall need some structural hypotheses on σ relative to the reproducing kernels.

Definition 3.3. *Let $q \in]1, \infty[$, we say that the measure σ verifies the structural hypothesis $SH(q)$ if, with q' the conjugate of q:*

$$\exists \alpha = \alpha_q > 0\ s.t.\ \forall a \in \mathcal{M}_q \cap \mathcal{M}_{q'} \subset \mathcal{M}_2,\ \|k_a\|_2^2 \geq \alpha \|k_a\|_q \|k_a\|_{q'} . \tag{3.1}$$

This is opposite to the Hölder inequalities.
Because $a \in \mathcal{M}_q \cap \mathcal{M}_{q'} \subset \mathcal{M}_2$, we have $k_a(a) = \displaystyle\int_X k_a(\zeta)\overline{k}_a(\zeta)\, d\sigma = \|k_a\|_2^2$ and the condition above is the same as

$$\|k_a\|_q \|k_a\|_{q'} \leq \alpha_q^{-1} k_a(a).$$

Definition 3.4. *Let $p, s \in [1, \infty]$ and q such that $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. We say that the measure σ verifies the structural hypothesis $SH(p, s)$ if*

$$\exists \beta = \beta_{p,q} > 0 \ s.t. \ \forall a \in \mathcal{M}_s,\ \|k_a\|_{s'} \leq \beta \|k_a\|_{p'} \|k_a\|_{q'}. \tag{3.2}$$

This is meaningfull because $s < p,\ s < q$ hence $\mathcal{M}_s \subset \mathcal{M}_p \cap \mathcal{M}_q$.

In the case of the unit ball $\mathbb{B} \subset \mathbb{C}^n$ and σ the Lebesgue measure on $X = \partial\mathbb{B}$ and in the case of the polydisc $\mathbb{D}^n \subset \mathbb{C}^n$ and σ the Lebesgue measure on $\mathbb{T}^n$, it is shown in [3] that these two hypotheses are verified for all p, s, q.

3.2. **Interpolating sequences.** We shall use the following facts proved in [3] :

Theorem 3.5. *If, for a $p \geq 1$, $S \subset \mathcal{M}_p$, if $S \in IH^\infty(\sigma)$ and if σ verifies $SH(p)$ then $S \in IH^p(\sigma)$ with the L.E.P..*

Theorem 3.6. *If $S \subset \mathcal{M}_1$ and S is dual bounded in $H^p(\sigma)$ for a $p > 1$, then $S \in IH^1(\sigma)$.*

We shall need to truncate S to its first N elements, say S_N. Clearly if $S \in IH^p(\sigma)$ then $S_N \in IH^p(\sigma)$ with a smaller constant than C_I. Let $I^p_{S_N} := \{f \in H^p(\sigma)\ s.t.\ f_{|S_N} = 0\}$ be the module over A of the functions zero on S_N. We have then for $\lambda \in \ell^p$, with $\{\rho_a\}_{a\in S}$ a bounded dual sequence, that the function $f_N := \sum_{a\in S_N} \lambda_a \rho_a$ interpolates λ on S_N and we have $\|f_N\|_{H^p(\sigma)/I^p_{S_N}} \leq C_I \|\lambda\|_p$.

We also have the converse for $1 < p \leq \infty$, which is all what we need [3]:

Lemma 3.7. *If S is such that all its truncations S_N are in $IH^p(\sigma)$ for a $p > 1$, with a uniform constant C_I then $S \in IH^p(\sigma)$ with the same constant.*

4. Carleson sequences.

As before we denote by $k_{q,a} := \dfrac{k_a}{\|k_a\|_q}$ the normalized reproducing kernel in $H^q(\sigma)$.

Definition 4.1. *We say that the sequence $S \subset \mathcal{M}_{q'}$ is a q-Carleson sequence if $1 \leq q < \infty$ and*

$$\exists D_q > 0,\ \forall \mu \in \ell^q,\ \left\|\sum_{a\in S} \mu_a k_{q,a}\right\|_q \leq D_q \|\mu\|_q.$$

We say that the sequence $S \subset \mathcal{M}_{q'}$ is a weakly q-Carleson sequence if $2 \leq q < \infty$ and

$$\exists D_q > 0,\ \forall \mu \in \ell^q,\ \left\|\sum_{a\in S} |\mu_a|^2 |k_{q,a}|^2\right\|_{q/2} \leq D_q \|\mu\|_q^2.$$

We call "weakly" Carleson the second condition because

Lemma 4.2. *If $2 \leq q < \infty$ and S is q-Carleson then it is weakly q-Carleson.*

Proof

for a sequence S we introduce a related sequence $\{\epsilon_a\}_{a\in S}$ of independent random variables with the same law $P(\epsilon_a = 1) = P(\epsilon_a = -1) = 1/2$. We shall denote by $\mathbb{E}$ the associated expectation.

Let S be a q-Carleson sequence, with the associated $\{\epsilon_a\}_{a\in S}$ we have

$$\left\|\sum_{a\in S}\mu_a\epsilon_a k_{q,a}\right\|_q^q \lesssim \|\mu\|_q^q$$

because $|\epsilon_a| = 1$. Taking expectation on both sides leads to

$$\left\|\mathbb{E}\left[\left|\sum_{a\in S}\mu_a\epsilon_a k_{q,a}\right|^q\right]\right\|_1 = \mathbb{E}\left[\left\|\sum_{a\in S}\mu_a\epsilon_a k_{q,a}\right\|_q^q\right] \lesssim \|\mu\|_q^q .$$

Now using Khintchine's inequalities for the left expression

$$\left\|\mathbb{E}\left[\left|\sum_{a\in S}\mu_a\epsilon_a k_{q,a}\right|^q\right]\right\|_1 \simeq \left\|\sum_{a\in S}|\mu_a|^2\,|k_{q,a}|^2\right\|_{q/2}^{q/2},$$

we get

$$\left\|\sum_{a\in S}|\mu_a|^2\,|k_{q,a}|^2\right\|_{q/2}^{q/2} \lesssim \mathbb{E}\left[\left\|\sum_{a\in S}\mu_a\epsilon_a k_{q,a}\right\|_q^q\right] \lesssim \|\mu\|_q^q,$$

and the lemma. □

Now if S is weakly p-Carleson is S weakly q-Carleson for other q ?

Notice that any sequence S is weakly 2-Carleson :

$$\forall\nu\in\ell^1,\ \left\|\sum_{a\in S}\nu_a\,|k_{2,a}|^2\right\|_1 \leq \sum_{a\in S}|\nu_a|\,\left\||k_{2,a}|^2\right\|_1 \leq \|\nu\|_1,$$

because $\|k_{2,a}\|_2 = \left\||k_{2,a}|^2\right\|_1 = 1$.

Hence if S is weakly q-Carleson with $q > 2$ we can try to use interpolation of linear operators.

Let us define our operator T :

$$T\ :\ \ell^q(\omega_q)\longrightarrow L^q(\sigma);\ T\lambda := \sum_{a\in S}\lambda_a\,|k_a|^2,$$

with the weight $\omega_q(a) := \|k_a\|_{2q}^{-2q}$; this means that

$$\lambda\in\ell^q(\omega_q)\Longrightarrow \|\lambda\|_{\ell^q(\omega_q)}^q := \sum_{a\in S}|\lambda_a|^q\,\omega_q(a) < \infty.$$

By a theorem of E. Stein and G. Weiss [17] we know that if T is bounded from $\ell^q(\omega_q)$ to $L^q(\sigma)$ and from $\ell^1(\omega_1)$ to $L^1(\sigma)$ then T is bounded from $\ell^p(\omega'_p)$ to $L^p(\sigma)$ with $1 \leq p \leq q$ provided that the weight satisfies the condition

$$\text{if } \frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{q} \text{ then } \omega'_p = \omega_1^{p(1-\theta)}\omega_q^{p\theta/q}.$$

Here this means

$$\omega'_p(a) = \|k_a\|_2^{-2p(1-\theta)}\,\|k_a\|_{2q}^{-2p\theta} .$$

Then $\|T\lambda\|_p^p \lesssim \|\lambda\|_{\ell^p(\omega'_p)}^p = \sum_{a\in S}|\lambda_a|^p\,\omega'_p(a)$. Hence if $\omega'_p(a) \lesssim \omega_p(a)$ we shall have

$$\|T\lambda\|_p^p \lesssim \|\lambda\|_{\ell^p(\omega'_p)}^p = \sum_{a\in S}|\lambda_a|^p\,\omega'_p(a) \lesssim \sum_{a\in S}|\lambda_a|^p\,\omega_p(a),$$

and this will be OK.

Lemma 4.3. *Let $q\geq 1$ and $\displaystyle\frac{1}{p}=\frac{1-\theta}{1}+\frac{\theta}{q}$ with $0<\theta<1$, then*

$$\|k_a\|_{2p}\leq \|k_a\|_2^{(1-\theta)}\,\|k_a\|_{2q}^{\theta}\,,$$

Proof

let $\displaystyle\frac{1}{p}=\frac{1-\theta}{1}+\frac{\theta}{q}=\frac{1}{s}+\frac{1}{r}$ with $\displaystyle s=\frac{1}{1-\theta}$ and $\displaystyle r=\frac{q}{\theta}$.

Hölder's inequalities give, for $f\in L^s(\sigma),\ g\in L^r(\sigma)$

$$\left(\int_X |fg|^p\,d\sigma\right)^{1/p}\leq\left(\int_X |f|^s\,d\sigma\right)^{1/s}\left(\int_X |g|^r\,d\sigma\right)^{1/r}.$$

Set $f=|k_a|^{2(1-\theta)}\,,\quad g:=|k_a|^{2\theta}$ we get

$$\left(\int_X |k_a|^{2p}\,d\sigma\right)^{1/p}\leq\left(\int_X |k_a|^{2(1-\theta)s}\,d\sigma\right)^{1/s}\left(\int_X |k_a|^{2\theta r}\,d\sigma\right)^{1/r},$$

hence replacing s,r

$$\left(\int_X |k_a|^{2p}\,d\sigma\right)^{1/p}\leq\left(\int_X |k_a|^{2}\,d\sigma\right)^{1-\theta}\left(\int_X |k_a|^{2q}\,d\sigma\right)^{\theta/q},$$

hence

$$\|k_a\|_{2p}\leq \|k_a\|_2^{(1-\theta)}\,\|k_a\|_{2q}^{\theta}\,,$$

and the lemma. $\square$

Back to our operator T, we have $\omega'_p(a)=\|k_a\|_2^{-2p(1-\theta)}\,\|k_a\|_{2q}^{-2p\theta}$ but the lemma above says $\|k_a\|_{2p}\lesssim\|k_a\|_2^{1-\theta}\,\|k_a\|_{2q}^{\theta}$ with $\displaystyle\frac{1}{p}=\frac{1-\theta}{1}+\frac{\theta}{q}$ which implies $\omega'_p(a)\lesssim\|k_a\|_{2p}^{-2p}=\omega_p(a)$ and the condition of the Stein-Weiss theorem are fullfilled, so we proved

Lemma 4.4. *If S is weakly q-Carleson, with $q>2$ then S is weakly p-Carleson for any $2\leq p\leq q$.*

We notice too that any sequence S is 1-Carleson

$$\forall\mu\in\ell^1,\ \left\|\sum_{a\in S}\mu_a k_{1,a}\right\|_1\leq\sum_{a\in S}|\mu_a|\,\|k_{1,a}\|_1\leq\|\mu\|_1,$$

and the same proof as above gives

Lemma 4.5. *If S is q-Carleson, with $q>1$ then S is p-Carleson for any $1\leq p\leq q$.*

In the ball or in the polydisc, we have much better:

Remark 4.6. *If S is q-Carleson for a $q\in]1,\infty[$ then S is p-Carleson for any p. Moreover S q-Carleson is equivalent to S weakly $2q$-Carleson.*

5. Main results

Now we are in position to state our main results.

Theorem 5.1. *Let $p\geq 1$, $1\leq s<p$ and q be such that $\displaystyle\frac{1}{s}=\frac{1}{p}+\frac{1}{q}$. Suppose that $S\subset\mathcal{M}_s\cap\mathcal{M}_{q'}$, that S is dual bounded in $H^p(\sigma)$, $p\leq 2.$, that S is weakly q-Carleson and σ verifies the structural hypotheses $SH(q)$ and $SH(p,s)$. Then S is $H^s(\sigma)$ interpolating and has the L.E.P. in $H^s(\sigma)$.*

Using this time the fact that Kinchine's inequalities also provide a way to put absolut values inside sums, we get the other extremity for the range of p's:

Theorem 5.2. *Let $1 \leq s < \infty$. Suppose that $S \subset \mathcal{M}_s \cap \mathcal{M}_{s'}$, that S is dual bounded in $H^\infty(\sigma)$, S is weakly p-Carleson for a $p > s$ and $(A,\ \sigma)$ verify the structural hypotheses $SH()$. Then S is $H^s(\sigma)$ interpolating with the L.E.P..*

These theorems will be consequence of the next lemma.

As above, if S is a sequence of points in $\mathcal{M}$, we introduce the related sequence $\{\epsilon_a\}_{a\in S}$ of independent Bernouilli variables.

Lemma 5.3. *Let $S \subset \mathcal{M}_p$ be a sequence of points such that a dual system $\{\rho_{p,a}\}_{a\in S}$ exists in $H^p(\sigma)$; let $1 \leq s < p$ and q be such that $\dfrac{1}{s} = \dfrac{1}{p} + \dfrac{1}{q}$;*

if $\forall \lambda \in \ell^p(S),\ \mathbb{E}\left[\left\Vert \sum_{a\in S} \lambda_a \epsilon_a \rho_{p,a} \right\Vert_p^p\right] \lesssim \Vert\lambda\Vert_{\ell^p}^p$, S is q-weakly Carleson and σ verifies $SH(q)$, $SH(p,s)$ then S is $H^s(\sigma)$ interpolating and moreover S has the L.E.P..

Proof

If $p = 1$ we have nothing to prove: the functions $\rho_{1,a}$ are uniformly bounded in $H^1(\sigma)$, just set

$$\forall \lambda \in \ell^1,\ T(\lambda) := \sum_{a\in S} \lambda_a \rho_{1,a},$$

this function interpolates the sequence λ, is bounded in $H^1(\sigma)$, and clearly the operator T is also linear and bounded.

If $p > 1$, we may suppose that $1 < s < p$ because if $S \in IH^s(\sigma)$ then by theorem 3.6, for $S \subset \mathcal{M}_1$ we also have that $S \in IH^1(\sigma)$.

First we truncate the sequence: S_N is the first N elements of S. We shall get estimates independent of N, i.e.

for $s \in [1, p[$ and $\nu \in \ell_N^s$ we shall built a function $h \in H^s(\sigma)$ such that:

$$\forall j = 0, ..., N-1,\ h(a_j) = \nu_j \left\Vert k_{a_j}\right\Vert_{s'} \text{ and } \Vert h\Vert_{H^s} \leq C\, \Vert\nu\Vert_{\ell_N^s},$$

with the constant C independent of N. We conclude then by use of lemma 3.7.

We choose q such that $\dfrac{1}{s} = \dfrac{1}{p} + \dfrac{1}{q}$; then $q \in]p', \infty[$ with p' the conjugate exponent of p and we set $\nu_j = \lambda_j \mu_j$ with $\mu_j := |\nu_j|^{s/q} \in \ell^q,\ \lambda_j := \frac{\nu_j}{|\nu_j|}|\nu_j|^{s/p} \in \ell^p$ then $\Vert\nu\Vert_s = \Vert\lambda\Vert_p \Vert\mu\Vert_q$.

Let $c_a := \dfrac{\Vert k_a\Vert_{s'}}{\Vert k_a\Vert_{p'}\, k_{q,a}(a)} = \dfrac{\Vert k_a\Vert_{s'} \Vert k_a\Vert_q}{\Vert k_a\Vert_{p'}\, k_a(a)}$. By $SH(q)$ we have $k_a(a) \geq \alpha \Vert k_a\Vert_q \Vert k_a\Vert_{q'}$ hence

$$c_a \leq \frac{\Vert k_a\Vert_{s'}}{\alpha^{-1}\Vert k_a\Vert_{p'}\Vert k_a\Vert_{q'}} \text{ and by } SH(p,s) \text{ we get } c_a \leq \alpha^{-1}\beta.$$

(i) Now set $h(z) = \sum_{a\in S} \nu_a c_a \rho_a k_{q,a}$ then:

$$h(a) = \nu_a \Vert k_a\Vert_{s'} \text{ because } \rho_a(b) = \delta_{ab} \Vert k_a\Vert_{p'}.$$

These are the good values, hence h interpolates ν and moreover h is clearly linear in ν.

(ii) Estimate on the $H^s(\sigma)$ norm of h.

Set

$$f(\epsilon, z) := \sum_{a\in S} \lambda_a c_a \epsilon_a \rho_a(z), \qquad g(\epsilon, z) := \sum_{a\in S} \mu_a \epsilon_a k_{q,a}(z).$$

Then $h(z) = \mathbb{E}(f(\epsilon, z) g(\epsilon, z))$ because $\mathbb{E}(\epsilon_j \epsilon_k) = \delta_{jk}$.

So we get

$$|h(z)|^s = |\mathbb{E}(fg)|^s \leq (\mathbb{E}(|fg|))^s \leq \mathbb{E}(|fg|^s),$$

hence

$$\|h\|_s = \left(\int_X |h(z)|^s \, d\sigma(z)\right)^{1/s} \leq \left(\int_X \mathbb{E}(|fg|^s) \, d\sigma(z)\right)^{1/s}.$$

But, using Hölder's inequality, we get

$$\int_X \mathbb{E}(|fg|^s) \, d\sigma(z) = \mathbb{E}\left[\int_X |fg|^s \, d\sigma(z)\right] \leq \left(\mathbb{E}\left[\int_X |f|^p \, d\sigma\right]\right)^{s/p} \left(\mathbb{E}\left[\int_X |g|^q \, d\sigma\right]\right)^{s/q}. \tag{5.1}$$

Let $\forall a \in S,\ \tilde{\lambda}_a := c_a\lambda_a \Longrightarrow \left\|\tilde{\lambda}\right\|_p \leq \alpha\beta \left\|\lambda\right\|_p$ and the first factor is controlled by the lemma hypothesis

$$\mathbb{E}\left[\int_X |f|^p \, d\sigma\right] = \mathbb{E}\left[\left\|\sum_{a\in S} \lambda_a c_a \epsilon_a \rho_{p,a}\right\|_p^p\right] \lesssim \left\|\tilde{\lambda}\right\|_p^p \lesssim \|\lambda\|_{\ell^p}^p. \tag{5.2}$$

Fubini theorem gives for the second factor

$$\mathbb{E}\left[\int_X |g|^q \, d\sigma\right] = \int_X \mathbb{E}\left[|g|^q\right] \, d\sigma.$$

We apply Khintchine's inequalities to $\mathbb{E}\left[|g|^q\right]$

$$\mathbb{E}\left[|g|^q\right] \simeq \left(\sum_{a\in S} |\mu_a|^2 \left|k_{q,a}\right|^2\right)^{q/2},$$

hence S being weak q-Carleson implies

$$\int_X \mathbb{E}\left[|g|^q\right] \, d\sigma \lesssim \int_X \left(\sum_{a\in S} |\mu_a|^2 \left|k_{q,a}\right|^2\right)^{q/2} d\sigma \lesssim \|\mu\|_{\ell^q}^q. \tag{5.3}$$

So putting (5.2) and (5.3) in (5.1) we get the lemma. □

5.1. **Proof of theorem 5.1.** Let us recall the theorem we want to prove.

Theorem 5.4. *Let $p \geq 1$, $1 \leq s < p$ and q be such that $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. Suppose that $S \subset \mathcal{M}_s \cap \mathcal{M}_{q'}$, that $\{\rho_{p,a}\}_{a\in S}$ is a norm bounded sequence in $H^p(\sigma)$, $p \leq 2$, that S is weakly q-Carleson and σ verifies the structural hypotheses $SH(q)$, $SH(p,s)$. Then S is $H^s(\sigma)$-interpolating with the L.E.P..*

It remains to prove that the hypotheses of the theorem implies those of the lemma 5.3.

We have to prove that

$$\mathbb{E}\left[\left\|\sum_{a\in S} \lambda_a \epsilon_a \rho_{p,a}\right\|_p^p\right] \lesssim \|\lambda\|_{\ell^p}^p,$$

knowing that the dual sequence $\{\rho_{p,a}\}_{a\in S}$ is bounded in $H^p(\sigma)$, i.e.

$$\sup_{a\in S} \left\|\rho_{p,a}\right\|_p \leq C.$$

By Fubini's theorem

$$\mathbb{E}\left[\left\|\sum_{a\in S} \lambda_a \epsilon_a \rho_{p,a}\right\|_p^p\right] = \int_X \mathbb{E}\left[\left|\sum_{a\in S} \lambda_a \epsilon_a \rho_{p,a}\right|^p\right] d\sigma,$$

and by Khintchine's inequalities we have

$$\mathbb{E}\left[\left|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right|^p\right]\simeq\left(\sum_{a\in S}\left|\lambda_a\right|^2\left|\rho_{p,a}\right|^2\right)^{p/2}.$$

Now $p\leq 2$, so $\left(\sum_{a\in S}\left|\lambda_a\right|^2\left|\rho_{p,a}\right|^2\right)^{1/2}\leq\left(\sum_{a\in S}\left|\lambda_a\right|^p\left|\rho_{p,a}\right|^p\right)^{1/p}$ hence

$$\int_X\mathbb{E}\left[\left|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right|^p\right]d\sigma\leq\int_X\left(\sum_{a\in S}\left|\lambda_a\right|^p\left|\rho_{p,a}\right|^p\right)d\sigma=\sum_{a\in S}\left|\lambda_a\right|^p\left\|\rho_{p,a}\right\|_p^p.$$

So, finally

$$\mathbb{E}\left[\left\|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right\|_p^p\right]\lesssim\sup_{a\in S}\ \left\|\rho_{p,a}\right\|_p^p\left\|\lambda\right\|_p^p,$$

and the theorem 5.1. $\square$

Suggested by F. Lust-Piquard, one can use that $H^p(\sigma)\subset L^p(\sigma)$ hence, because $p\leq 2$, $H^p(\sigma)$ is of type p which means precisely([13], Th III.9) that $\mathbb{E}\left[\left|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right|^p\right]\lesssim\sum_{a\in S}\left|\lambda_a\rho_a\right|^p$, hence integrating and using Fubini, we get

$$\mathbb{E}\left[\left\|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right\|_p^p\right]\lesssim\int\sum_{a\in S}\left|\lambda_a\rho_a\right|^p d\sigma\lesssim(\sup_{a\in S}\ \left\|\rho_{p,a}\right\|_p)\left\|\lambda\right\|_{\ell^p}^p\lesssim\left\|\lambda\right\|_{\ell^p}^p.$$

And again the theorem.

5.2. **Proof of theorem 5.2.** Let us recall the theorem we want to prove.

Theorem 5.5. *Let $1\leq s<\infty$. Suppose that $S\subset\mathcal{M}_s\cap\mathcal{M}_{s'}$, that $\{\rho_a\}_{a\in S}$ is a norm bounded sequence in $H^\infty(\sigma)$, weakly p-Carleson for a $p>s$ and σ verifies the structural hypotheses $SH(p,s)$, $SH(q)$ for q such that $\frac{1}{s}=\frac{1}{p}+\frac{1}{q}$. Then S is $H^s(\sigma)$ interpolating with the L.E.P..*

Proof
the idea is still to use lemma 5.3, but in two steps. Let $s<\infty$ be given and take p such that $s<p<\infty$ and S is weakly p-Carleson.

Set $\forall a\in S,\ \rho_{p,a}:=\rho_a k_{p,a}$. We have $\left\|\rho_{p,a}\right\|_p\leq\left\|\rho_a\right\|_\infty\left\|k_{p,a}\right\|_p=\left\|\rho_a\right\|_\infty\leq C$ by hypothesis. We want to prove that

$$\mathbb{E}\left[\left\|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right\|_p^p\right]=\int_X\mathbb{E}\left[\left|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right|^p\right]d\sigma\lesssim\left\|\lambda\right\|_{\ell^p}^p,$$

in order to apply lemma 5.3.

By Khintchine's inequalities we have

$$\mathbb{E}\left[\left|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right|^p\right]\simeq\left(\sum_{a\in S}\left|\lambda_a\right|^2\left|\rho_{p,a}\right|^2\right)^{p/2},$$

but this time we use that $\left|\rho_{p,a}\right|\leq\left\|\rho_{\infty,a}\right\|\left|k_{a,p}\right|\leq C\left|k_{a,p}\right|$ hence

$$\mathbb{E}\left[\left|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right|^p\right]\lesssim C^p\left(\sum_{a\in S}\left|\lambda_a\right|^2\left|k_{a,p}\right|^2\right)^{p/2}.$$

Using that S is weakly p-Carleson, we get

$$\left\|\sum_{a\in S}\left|\lambda_a\right|^2\left|k_{a,p}\right|^2\right\|_{p/2}^{p/2}\leq D\left\|\lambda\right\|_p^p,$$

hence

$$\mathbb{E}\left[\left\|\sum_{a\in S}\lambda_a\epsilon_a\rho_{p,a}\right\|_p^p\right]\lesssim C^p\int_X\left(\sum_{a\in S}\left|\lambda_a\right|^2\left|k_{a,p}\right|^2\right)d\sigma\lesssim DC^p\left\|\lambda\right\|_{\ell^p}^p,$$

and we can apply the lemma 5.3 which gives the theorem because $p > s$ implies that S is still weakly s-Carleson by lemma 4.4. □

6. Application to the ball and to the polydisc.

In [3] it is proved that the structural hypotheses hold in the polydisc. Moreover the Carleson measures, hence the Carleson sequences, are characterized geometrically and they are the same for all $p\in]1,\infty[$ (see [8], [9]). So it is enough to say "Carleson sequence" in the theorem:

Theorem 6.1. *Let $S\subset\mathbb{D}^n$ be a Carleson sequence and dual bounded in $H^p(\mathbb{D}^n)$ with either $p=\infty$ or $p\leq 2$, then S is $H^s(\mathbb{D}^n)$ interpolating for any $s<p$ with the LEP.*

Still in [3] it is proved that the structural hypotheses hold in the ball. Again the Carleson measures, hence the Carleson sequences, are characterized geometrically and they are the same for all $p\in]1,\infty[$ (see [11])but moreover a theorem of P. Thomas [18] gives that S dual bounded in $H^p(\mathbb{B})$ implies S Carleson, hence

Theorem 6.2. *Let $S\subset\mathbb{B}$ be dual bounded in $H^p(\mathbb{B})$ with either $p=\infty$ or $p\leq 2$, then S is $H^s(\mathbb{B})$ interpolating for any $s<p$ with the LEP.*

We have for free the same result for the Bergman classes of the ball by the "subordination lemma" [1]:

to a function $f(z)$ defined on $z=(z_1,...,z_n)\in\mathbb{B}_n\subset\mathbb{C}^n$ associate the function.

$\tilde{f}(z,w):=f(z)$ defined on $(z,w)=(z_1,...,z_n,w)\in\mathbb{B}_{n+1}\subset\mathbb{C}^{n+1}$.

Then we have that $f\in A^p(\mathbb{B}_n)\Longleftrightarrow\tilde{f}\in H^p(\mathbb{B}_{n+1})$ with the same norm. Moreover if $F\in H^p(\mathbb{B}_{n+1})$then $f(z):=F(z,0)\in A^p(\mathbb{B}_n)$ with $\left\|f\right\|_{A^p(\mathbb{B}_n)}\leq\left\|F\right\|_{H^p(\mathbb{B}_{n+1})}$.

Suppose that $S\subset\mathbb{B}_n$ is dual bounded in $A^p(\mathbb{B}_n)$ this means that

$\exists\{\rho_a\}_{a\in S}$ *s.t.* $\forall a\in S,\left\|\rho_a\right\|_{A^p(\mathbb{B}_n)}\leq C$ and $\rho_a(b)=\delta_{ab}(1-|a|^2)^{-(n+1)/p}$,

because the normalized reproducing kernel for $A^p(\mathbb{B}_n)$ is $b_a(z):=\dfrac{(1-|a|^2)^{(n+1)/p'}}{(1-\overline{a}\cdot z)^{n+1}}$.

Embed S in $\mathbb{B}_{n+1}$ by $\tilde{S}:=\{(a,0),\ a\in S\}$ as in [1], then the sequence $\{\tilde{\rho}_a\}_{a\in S}$ is precisely a bounded dual sequence for $\tilde{S}\subset\mathbb{B}_{n+1}$ in $H^p(\mathbb{B}_{n+1})$ hence we can apply the previous theorem:

if $p=\infty$ or $p\leq 2$ then $\tilde{S}$ is $H^s(\mathbb{B}_{n+1})$ interpolating with the *L.E.P.*. If T is the operator making the extension,

$$\forall\lambda\in\ell^s\longrightarrow T\lambda\in H^s(\mathbb{B}_{n+1}),\ (T\lambda)(a,0)=\lambda_a\left\|k_{(a,0)}\right\|_{H^{s'}(\mathbb{B}_{n+1})},\ \left\|T\lambda\right\|_{H^s(\mathbb{B}_{n+1})}\leq C_I\left\|\lambda\right\|_s$$

then the operator $(U\lambda)(z) := (T\lambda)(z,0)$ is a bounded linear operator from ℓ^s to $A^s(\mathbb{B}_n)$ making the extension because $\left\|k_{(a,0)}\right\|_{H^{s'}(\mathbb{B}_{n+1})} = \left\|b_a\right\|_{A^{s'}(\mathbb{B}_b)}$ where k is the kernel for $H^s(\mathbb{B}_{n+1})$ and b is the kernel for $A^s(\mathbb{B}_n)$. Hence we proved

Corollary 6.3. *Let $S \subset \mathbb{B}$ be dual bounded in $A^p(\mathbb{B})$ with either $p = \infty$ or $p \leq 2$, then S is $A^s(\mathbb{B})$ interpolating for any $s < p$ with the LEP.*

We also get the same result for the Bergman spaces with weight of the form $(1 - |z|^2)^k,\ k \in \mathbb{N}$ just by the same method, but considering $H^p(\mathbb{B}_{n+k+1})$ instead of $H^p(\mathbb{B}_{n+1})$.

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